**Stability Theory**

**Stability:**  The solution of the differential equation is called stable if for each , there exists a such that for any solution of .

for all .

**Unstable:**  The solution of the differential equation is called unstable if it is not stable.

**Asymptotic stability:** The solution of the differential equation is called asymptotically stable if it is stable and if there exists a such that for any solution of

.

**Uniform stability**: The solution of the differential equation is called uniformly stable if for each there exists a such that for any solution of , the inequalities

and imply for all .

**Question-01:** Prove that all solutions of where is an continuous matrix in and is an vector, are stable if and only if they are bounded.

**Answer:** First we suppose that all solutions of

are bounded. Let be a solution of (1) passing through the point . We shall show that it is stable, that is, for any solution of (1), we have to show that for all , there exists a such that the inequality

holds.

Let be a fundamental matrix of (1). Then we have

and

since all solutions of (1) are bounded, so there exists a constant such that

Now

If we choose , then

Here and are both positive, so .

Thus, for all there exists a such that for any solution of (1) we have

Therefore all solutions of (1) are stable.

Conversely, suppose that all solutions of (1) are stable. We have to show that they are bounded.

Since all solutions of (1) are stable, so its zero solution is also stable, that is for all , there exists a such that

that is,

We know that and hence from (3) we have

Let be a vector with in the ith place and zero elsewhere.

Then from (4) we have

, where is the ith column of .

Therefore, , for all ;

Thus,

Hence all solutions of (1) are bounded. (**Proved**)

**Question-02:** If all the characteristic roots of have negative real parts, then every solution of where is a constant matrix, is asymptotically stable.

**Solution:** Given that

where is a constant matrix.

Since all the characteristic roots of have negative real parts, so there exists positive constants and such that

, for all

where is a fundamental matrix of (1) satisfying . If and be two solutions of (1) with initial values and at , then we have

and

First we show that the solutions of (1) are stable and next we show that they are asymptotically stable.

For stability we need to show that such that

, for all holds.

Now

.

If we select , then

Therefore all solutions of (1) are stable.

Also from (3) we have

Hence every solution of (1) is asymptotically stable. (**Proved**)

**Question-03:** Let be a fundamental matrix of the system such that Then prove that the system is asymptotically stable iff as .

**Solution:** Given that

Let be a fundamental matrix of the system (1) such that

First suppose that all solutions of (1) are asymptotically stable.

We have to show that as .

Since all the solutions of (1) are asymptotically stable, its zero solution is asymptotically stable.

That is, as

as

as

as since

Conversely, suppose that as

We have to show that all the solutions of (1) are asymptotically stable.

Let be a solution of (1) with .

Then

Since is continuous, it follows that every solution of (1) is bounded and hence it is stable.

From (2) we have

as

Since as

as .

Hence all solutions of (1) are asymptotically stable. (**Proved**)

**Question-04:** Show that zero solution of is asymptotically stable iff and is uniformly stable iff is bounded above for

**Solution** Given that

1. First we suppose that the solution of (1) is asymptotically stable. We have to show

that .

Now from the definition of asymptotically stable we have, there exists such that

Conversely, suppose that

We have to show that the solution of (1) is asymptotically stable.

That is, there exists such that

.

Since

which implies that the zero solution is asymptotically stable.

Hence solution of (1) is asymptotically stable if and only if

.

1. We assume that the zero solution of (1) is uniformly stable.

We shall show that is bounded above for From the definition of uniform stability we have, for all there exists a such that for and

for all

for all

where

, for all

, where

is bounded above.

Conversely, suppose that is bounded above. We have to show that the solution of (1) is uniformly stable. For this we prove that, for all there exists a such that for we have

.

Since is bounded above, so there exists such that

,

,

, say

Now,

where

Thus, for we have a such that for ,

Hence the solution of (1) is uniformly stable.

**Problem**

**Problem-01:** Determine if system where , is stable, asymptotically stable.

**Solution:** Given that

where and

Let be the eigen value. Then the characteristic matrix of is

The characteristic polynomial of is,

=

=

The characteristic equation of is

Since all values of are negative, so the solutions are asymptotically stable and as such they are also stable. Thus the solutions of the given system are not unstable.

**Problem-02:** Test the asymptotically stability of the system

t)

**Solution:** Given that

where

Let be the eigen value. Then the characteristic matrix of is

The characteristic polynomial of is,

=

=

=

=

The characteristic equation of is

Since all real parts of are not negative, so the solutions are not asymptotically stable.

**Problem-03:** Prove that the zero solution of

(i) is uniformly stable but not asymptotically stable and

(ii) is asymptotically stable, but not uniformly stable.

**Solution: (i)** Given that

Let and

Then

and

The characteristic equation is

Therefore the eigen functions (solutions) are and .

The wronskian is

Therefore the solutions are linearly independent.

The fundamental matrix is

Therefore the zero solutions are uniformly stable.

Also

Hence the zero solutions are not asymptotically stable.

**(ii)** Given that

Let (t) be a solution of (1). Then we have

At we have

Putting this value in (2) we get the general solution of (1) is

Also the zero solution of (1) is .

To show that the zero solution is asymptotically stable, we first show that it is stable and then show that it is asymptotically stable.

To show stable, we have to show that

such that

Now

Taking we get,

Therefore any solution of (1) is stable.

For asymptotically stable we have

Therefore any solution of (1) is asymptotically stable.

**2nd part:** In this case we have to show that the zero solution of (1) is not uniformly stable.

Let , where or positive integer.

Also let , where or positive integer.

From (3) we have

and

Now

Since , so we have

Therefore the solution is not uniformly stable.

**Problem-04:** Determine the stability, the asymptotic stability or the instability of the system

**Solution:** Given that

Where

The characteristic matrix of is

The characteristic polynomial of is,

=

The characteristic equation of is

Since all values of are not negative, so the solutions are not asymptotically stable and hence not stable. Thus the solutions of the given system are unstable.

**Problem-05:** Show that the zero solution of is uniformly stable but not asymptotically stable.

**Solution:** Given that

Let and

Then

and

The characteristic equation is

Therefore the eigen functions (solutions) are and .

The wronskian is

Therefore the solutions are linearly independent.

The fundamental matrix is

Therefore the zero solutions are uniformly stable.

Also

Hence the zero solutions are not asymptotically stable. (**Showed**)

**Problem-06:** Determine whether each solution of the differential equation

is stable, asymptotically stable or unstable.

**Solution:** Given that

where and

Let be the eigen value. Then the characteristic matrix of is

The characteristic polynomial of is,

=

=

The characteristic equation of is

Since the real characteristic root and the real parts of complex characteristic roots of are negative, so the solutions are asymptotically stable and as such they are also stable. Thus the solutions of the given system are not unstable.